A Simple Method of Tuning PI Controllers for Unstable Systems with a Zero

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A simple method is proposed to design PI controllers for unstable first order plus time delay systems with a zero. The method is based on (i) matching the coefficient of corresponding first power of *s* in the numerator and that in the denominator of the closed loop transfer function for a servo problem and (ii) by specifying the initial (inverse) jump. This method gives simple equations for controller settings in terms of model parameters. Simulation results are given for robust performance of the controller for uncertainty in the value of the unstable pole and zero. The performance of the controller is evaluated by simulation on a CSTR with non-ideal mixing carrying out an enzymatic reaction.

Keywords:
PID controller, unstable system, zero

Introduction

Methods of designing PI controllers for stable FOPTD model are based on stability analysis,[1,2] constant open loop transfer function,[3] pole placement method,[4] stable inverse of the model[5] and synthesis method.[6] The design of PI controller for unstable FOPTD model has attracted attention recently.[7–10] The performance specifications that are normally obtained for stable FOPTD model cannot be obtained for unstable systems. The methods for designing PID controllers for unstable FOPTD systems are given by the modified Ziegler – Nichols method,[7,8,11] IMC method,[12] pole placement method,[13] Optimization method,[10,14] two degrees of freedom method[15,16] and synthesis method.[17] In all the above procedures, the design methods are somewhat complicated. Recently, a simple method is proposed by *Chidambaram* et al.[18] to design PID controller for stable FOPTD by equating the coefficient of the corresponding powers of *s* in the numerator and that in the denominator of the closed loop transfer function for a servo problem. Performance of the controller designed by this method is shown to be similar to that of Ziegler-Nichols method.

Since the performance specifications for stable systems can not be met for the unstable systems, *Chidambaram* et al.[18] have used one tuning parameter, α (i.e. each term in the numerator is equal to α times that of the denominator). The performance of the controller designed by the method is significantly better than that of pole placement method. Later Chidambaram and Padma Sree[19] have extended the method to integrating system with dead time, and the performance of the controller designed is significantly better than that of the optimization method proposed by Visioli[14].

The performance of the controller is limited by the presence of an unstable zero. Methods of designing PI controllers for stable systems with unstable zeros are available.[20,21] The closed loop response of such systems show large initial inverse response. In the present work, the simple method[18] is extended to design PI controllers for unstable systems with a zero. The transfer function with unstable pole and unstable zero occurs in modeling of enzymatic reaction in CSTR with Cholette's non-ideal mixing model,[22] in modeling of fluid catalytic reactors[23] and in modeling a CSTR for carrying out auto-catalytic reaction. Since the system with a zero shows an initial (inverse response in case of an unstable zero) jump, it is proposed in the present controller design method to use this value, and it is also proposed to match coefficient of power of *s* in numerator with that of denominator of closed loop transfer function for a servo problem.

The proposed method

Unstable system with an unstable zero

Let us consider a unstable first order system with a positive zero [$k_p\, G_p = k_p\, (1 - p\, s)/(\tau\, s - 1)$]. Let us use a PI controller. The closed loop transfer function relating the output variable (*y*) to the set point (y_r) is given by

$$y/y_r = k_c\, k_p\, (1 - p\, s)\, (\tau_I\, s + 1)/[\tau_I\, s\, (\tau\, s - 1) + k_c\, k_p\, (1 - p\, s)\, (\tau_I\, s + 1)] \quad (1)$$

$y(t)$ at $t = 0$ gives the value of the inverse jump. From the initial value theorem, we know that $y(t)$ at $t = 0$ can be obtained from the limiting value of value of $[s\ y(s)]$ as s tends to infinity.

$$[s\ y(s)] = -\ [k_c\ k_p\ p\ /(\tau - k_c\ k_p\ p)] = -\ \phi \quad (2)$$
$$Lt\ s \rightarrow \infty$$

From eq (2), we get

$$k_c = \phi\ \tau\ /\ [(1 + \phi)\ k_p\ p] \quad (3)$$

The overshoot and settling time for systems with positive zero is large. Hence, the numerator term of the coefficient of s equals to α times that of the corresponding denominator term. By doing so, we get

$$k_c\ k_p\ (\tau_I - p) = \alpha\ [-\tau_I + k_c\ k_p\ (\tau_I - p)] \quad (4)$$

From eq(4), we get

$$\tau_I = k_c\ k_p\ p\ (1 - \alpha)/[k_c\ k_p\ (1 - \alpha) + \alpha] \quad (5)$$

For an unstable system we can specify the value for ϕ. The limits of ϕ and α can be obtained by using Routh array stability criteria for the characteristic equation of the system as $\phi > 0$

$$\alpha > [\phi\ \tau/\ (\phi\ \tau - \phi\ p - p)] \quad (6)$$

For stability ϕ should be greater than 0. But if the initial jump is allowed to be small, the overshoot will be large. Basically we have to compromise between initial jump and overshoot. We can get the starting value of ϕ from eq. (3) as $\phi = [M / (1 - M)]$ where $M = k_c\ k_p\ (p/\tau)$. Taking the limiting value of $k_c\ k_p = 1$, we get $\phi = [p / (\tau - p)]$. Therefore, the starting value of ϕ should be greater than $[p / (\tau - p)]$ for unstable systems. α value is tuned greater than the value obtained from the RHS of eq (6). The value of ϕ dictates the value of k_c and the value of τ_I depends both on ϕ and α.

Unstable FOPTD system and with an unstable zero

Let us consider unstable first order plus time delay system with a positive zero. The transfer function of the process is given by $k_p\ G_p = k_p\ (1 - p\ s)\ e^{-Ls}\ /\ (\tau\ s - 1)$. Let us use a PI controller. The closed loop transfer function relating the output variable (y) and set point (y_r) is given by

$$y/y_r = k_c\ k_p\ (1 - p\ s)\ (\tau_I s + 1)\ e^{-Ls}/[\tau_I s\ (\tau\ s - 1)$$
$$+ k_c\ k_p\ (1 - p\ s)\ (\tau_I s + 1)\ e^{-Ls}] \quad (7)$$

In the above equation we shall remove $\exp(-Ls)$ term in the numerator for further analysis, since this will only shift the corresponding time axis. Using Pade's approximation for $\exp(-Ls)$ in the denominator, the order of the numerator is same as that of the denominator. $y(t')$ at $t' = L$ gives the value of the inverse jump. From the initial value theorem, we know that $y(t')$ at $t = L$ can be obtained from the limiting value of value of $[s\ y(s)]$ as s tends to infinity.

$$[s\ y(s)] = -\ [k_c\ k_p\ p\ /\tau] = -\ \phi \quad (8)$$
$$Lt\ s \rightarrow \infty$$

From eq (8), we get

$$k_c = \phi\ \tau\ /\ [k_p\ p] \quad (9)$$

By equating the coefficient of powers of s of the numerator with that of α times the denominator we get the following equation.

$$k_c k_p(\tau_I - p + 0.5\,L) = \alpha\ [-\tau_I + k_c k_p(\tau_I - p - 0.5\,L)] \quad (10)$$

From eq (10), we get

$$\tau_I = k_c k_p[p(1-\alpha) - 0.5\,L\,(1+\alpha)]/[k_c k_p(1-\alpha) + \alpha] \quad (11)$$

The limits of ϕ and α can be obtained by using Routh array stability criteria for the characteristic equation of the system as $\phi > 0$

$$\alpha > [\phi\ \tau/\ (\phi\ \tau - p)] \quad (12)$$

From the analysis of the closed loop system it can be shown that the initial jump is less for system with delay than system without delay. From the stability analysis ϕ should be greater than zero. It is suggested to get the starting value of ϕ as follows. From Eq.(9), we get $\phi = k_c\ k_p\ (p/\tau)$. Using the limiting value of $k_c\ k_p = 1$, $\phi = (p/\tau)$. Therefore, the starting value of ϕ should be greater than (p/τ) for unstable systems. α is tuned greater than the value α obtained from the RHS of eq(12).

Similar analysis for a stable zero gives the following conditions and equations for controller settings as shown in Table 1. System with a stable zero gives a positive initial jump and the overshoot is large.

Set point weighted PI controller

Systems with unstable poles with a stable zero give a large overshoot. Stable systems with unstable zero gives a large initial jump/undershoot. Use of a set point weighted PI controller reduces the overshoot and the initial jump. The PI control law for set point weighting parameter is given by

$$u(t) = k_c\ [(\beta\ y_r - y) + (1/\tau_I\ s)\ \smallint e\ dt] \quad (13)$$

In the present work, method of calculation of setpoint weighting parameter (β) for unstable system with a zero is proposed by extending the method suggested by Chidambaram[24] for the sys-

T a b l e 1 – *Conditions and equations for PI settings for unstable system with a stable zero*

System	Transfer function model	Conditions	Equation for PID settings
Unstable first order system with a stable zero	$k_p (1 + p s)/(\tau s - 1)$	$1 < \phi < 0$ (T1) $[\phi \tau / (\phi \tau + \phi p - p)] < \alpha < 1$ (T2)	$k_c = \phi\tau/[(1 - \phi) k_p p]$ (T5) $\tau_I = k_c k_p p (\alpha - 1)/[k_c k_p (1 - \alpha) + \alpha]$ (T6)
Unstable FOPTD system with a stable zero	$k_p (1 + p s) e^{-Ls} / (\tau s - 1)$	$\phi > 0$ (T3) $\alpha > [\phi \tau / (\phi \tau - p)]$ (T4)	$k_c = \phi \tau / [k_p p]$ (T7) $\tau_I = k_c k_p [p (1 - \alpha) +$ $+ 0.5 L (1 + \alpha)]/[k_c k_p (\alpha - 1) - \alpha]$ (T8)

T(1) – T(8) are referred to as equation numbers.

tem without zero. For the systems without any zero, Chidambaram [24] has derived equations for β as

$$\beta \tau_I = \tau_{e,c} \zeta \qquad (14)$$

where ζ is the damping coefficient and $\tau_{e,c}$ is reciprocal of the natural frequency of the closed loop system. For the system without any zero, the numerator of the closed loop system contains only the term $(\beta \tau_I + 1)$. For the systems with a zero the numerator consists of two terms $(\beta \tau_I + 1)$ and $(1 \pm ps)$. [+ sign for a stable zero and – sign for an unstable zero]. The two terms can be combined as $[(\beta \tau_I \pm p)s + 1]$. It can be easily shown from the work of Chidambaram[25] that the following equation holds good:

$$(\beta \tau_I \pm p) = \tau_{e,c} \zeta \qquad (15)$$

Hence, the value of setpoint weighting parameter is calculated as:

$$\beta = (\tau_{e,c} \zeta + p) / \tau_I \text{ (for an unstable zero)} \quad (16a)$$

$$\beta = (\tau_{e,c} \zeta - p) / \tau_I \text{ (for a stable zero)} \quad (16b)$$

Since $\tau_{e,c}$ and ζ can be obtained by dominant roots, it is thus easy to calculate the set point weighting parameter.

Simulation results

Case study 1

Let us consider a first order unstable system with a positive zero. $k_p = 1$, $\tau = 1$ and $p = 0.5$. For a value of $\phi = 2$ and the value of α from the RHS of eq (6) is 4. α value is varied as 4.4, 4.8 and 5.2. The corresponding value of k_c [from eq (3)] is 1.333 and τ_I [from eq (5)] are 17, 9.5 and 7 respectively. Servo response of the system is shown in Fig. 1. Since the initial jump and overshoot are significantly large, setpoint weighted PI controller is used. Set point weighting parameter is calculated from eq (16a) as $\beta = 0.1841$. Fig 1 shows that the set point weighted PI controller reduces the initial jump (from – 2 to – 0.2)
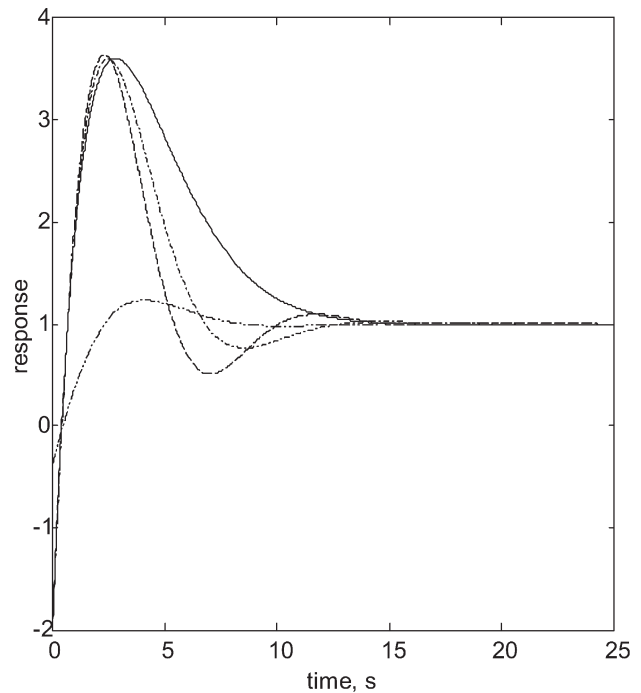

F i g . 1 – *Servo response of the system $(1 – 0.5 s)/(s – 1)$: k_c = 1.333*
Solid τ_I = 17, dot : τ_I = 9.5, dash : τ_I = 7, dot-dash : τ_I = 9.5 β = 0.1841

and overshoot (from 2.6 to 1.2) significantly. The PI controller is designed for nominal value of p. Whereas, while simulating we use + 20 % or – 20 % perturbation in p. Fig. 2 shows the robust performance. Similar response is obtained for uncertainty in model parameters τ and separately in k_p also. Since the system under consideration cannot be stabilized by a proportional controller, standard Ziegler – Nichols type tuning method cannot be used to compare the performance of the proposed method.

Case study 2

Let us consider a first order unstable system with a positive zero and with a delay. $k_p = 1$, $\tau = 1$, $p = 0.25$ and $L = 0.25$. For a value of $\phi = 0.4$, the value of α obtained from the RHS of eq (12) is
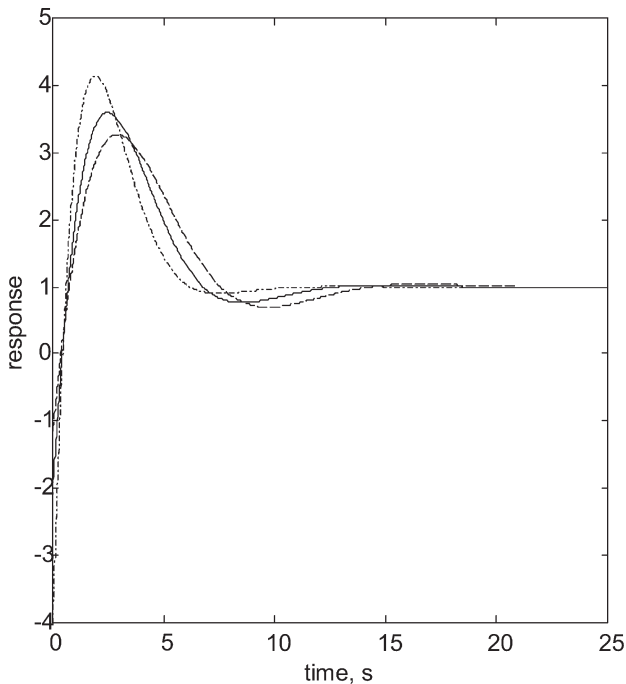
F i g . 2 – *Robustness of the controller for uncertainty in p for the system (1 − 0.5 s)/(s − 1). k_c = 1.333 τ_I = 9.5 solid : p = 0.5, dot : p = 0.6, dash : p = 0.4*

2.67. α value is varied as 2.72, 2.8 and 2.933. The corresponding value of k_c [from eq (9)] is 1.6 and τ_I [from eq (11)] are 44.75, 18.5 and 9.75 respectively. Servo response of the system is shown in Fig 3. Set point weighted PI controller [β = 0.1718 obtained
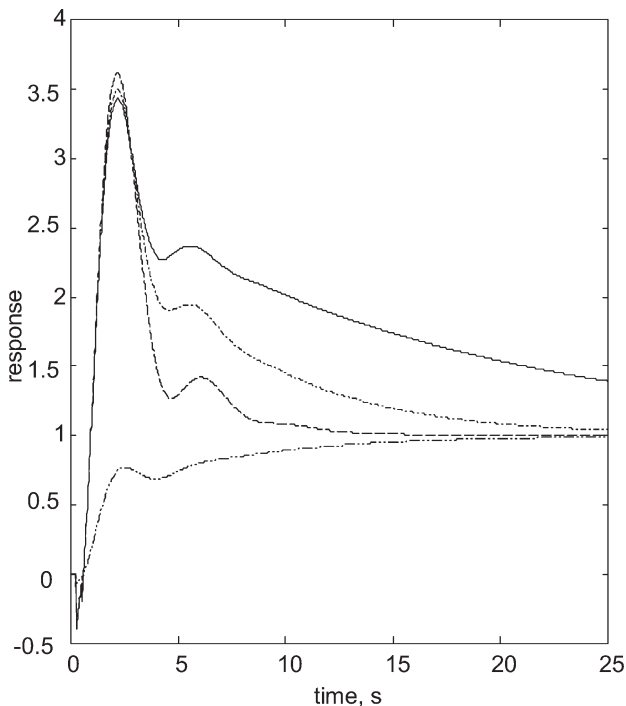

F i g . 3 – *Servo response of the system (1 − 0.25 s) $e^{-0.25 s}$/(s − 1) : k_c = 1.6 Solid : τ_I = 44.5, dot: τ_I = 18.5, dash : τ_I = 9.75, dot-dash : τ_I = 18.5 and β = 0.1718*

from eq (16a)] significantly reduces undershoot (from 0.4 to 0.045) and overshoot (from 2.65 to 0), as shown in Fig. 3. Robustness of the controller under parameter uncertainty in *p*, is shown in Fig. 4. Similar results are observed for uncertainty in parameters τ, k_p and *L* separately.
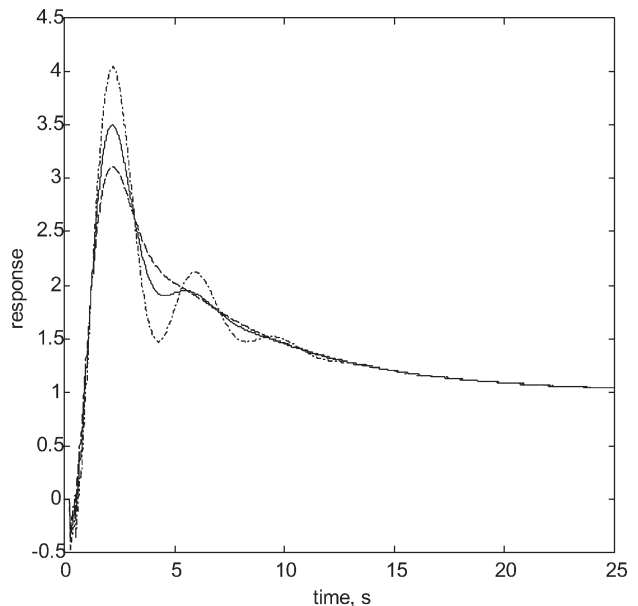

F i g . 4 – *Robustness of the controller for uncertainty in p for the system (1 − 0.25 s) $e^{-0.25s}$/(s − 1) k_c = 1.6 τ_I = 18.5Solid : p = 0.25, dot : p = 0.3, dash: p = 0.2*

Case study 3: Application to a CSTR with non-ideal mixing

Let us consider an isothermal CSTR with the reaction rate given by [$−k_1 c/(1 + k_2 c)^2$]. The non-ideal mixing is described by Cholette's model. Here *n* is the fraction of reactant feed that enters the zones of the perfect mixing and *m* is the fraction of total volume of the reactor where reaction occurs [i.e., (1 − *m*) fraction of the volume is dead zone]. The transient equation for the reactor is given by *Liou* and *Chien*[25]

$$dc/dt = (nQ/mV) (c_f − c) − [k_1 c/ (1 + k_2 c)^2] \qquad (17)$$

$$nc + (1 − n) c_f = c_e, \text{ at } t = 0, c = c_0 \qquad (18)$$

Here *c* is the concentration of the reactant in the well mixed reactor zone and c_e is the concentration of the reactant in the exit stream. The controlled variable is c_e and manipulated variable is the feed concentration c_f. For the present simulation study, we consider *n* = *m* = 0.75, k_1 = 10 s^{-1}, k_2 = 10 kmol m^{-3}, *V* = 10^{-3} dm^3.

This particular rate form [$−k_1 c /(1 + k_2 c)^2$] has been extensively studied[26] and its applicability to

heterogeneous and enzyme catalyzed reactions has been demonstrated. For $c_f = 3.288$ kmol m^{-3}, we get $c_e = 1.8$ kmol m^{-3} and $c = 1.304$ kmol m^{-3} and linearization of nonlinear equation around this nominal operating point gives the transfer function model as $\Delta c_e(s)/\rho c_f(s) = 2.21(1 + 11.133\ s)/(98.32\ s - 1)$. We have assumed a measurement delay of 20 sec in the derivation of above transfer function model. PI controller designed by using simple method with $\phi = 0.3$ and $\alpha = 1.2$ times the value obtained from eq (T4) (i.e. 1.93) gives $k_c = 1.2$ and $\tau_I = 94.74$. The performance of the controller on the nonlinear system for a step change in c_e from 1.8 to 1.9 is shown in Fig. 5. Set point weighting parameter calculated by eq (16b) is 0.1517. Fig. 5 shows that set point weighted PI controller reduces the overshoot significantly. The performance of the controller for regulatory problem (Q is changed from $0.03333 \cdot 10^{-3}$ to $0.3366\ 10^{-3}$ m^3 s^{-1}) is shown in Fig 6.

Conclusions

A simple method is proposed for PI settings for an unstable FOPTD system with a zero. The proposed method is used to design controller for various transfer function models and also applied to design controller for non-ideal CSTR, carrying out a reaction having reaction rate of the form $[-k_1\ c\ /(1 + k_2\ c)^2]$. Set point weighted PI controller significantly reduces undershoot and overshoot. The present method is robust for uncertainty in the model parameters.

Nomenclature and units

c – Reactant concentration in the reactor, kmol m^{-3}

c_e – Reactant concentration at the reactor exit, kmol m^{-3}

c_f – Reactant concentration in the feed, kmol m^{-3}

k_1 – Reaction rate coefficient, s^{-1}

k_2 – Constant, m^3 kmol^{-1}

k_c – Controller gain

k_p – Process gain

L – Time delay, s

m – Fraction of the total volume of the reactor which is perfectly mixed, dimensionless

n – Fraction of the feed entering the zone of perfect mixing, dimensionless

p – inverse of process zero, s

Q – Feed flow rate, m^{3}s^{-1}

t – Time, s

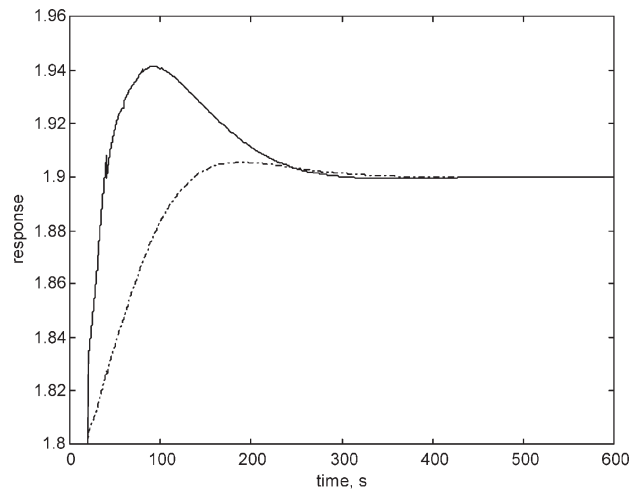V – Volume of the reactor, m^3

y – Output

F i g . 5 – *Servo response of the non-ideal CSTR carrying out a reaction with rate $[-k_1\ c\ /(1 + k_2\ c)^2]$ $k_c = 1.2$, $\tau_I = 94.74$ ($\phi = 0.3$ and $\alpha = 1.93$) solid: $\beta = 1$ (no set point weighting); chain: $\beta = 0.1517$*

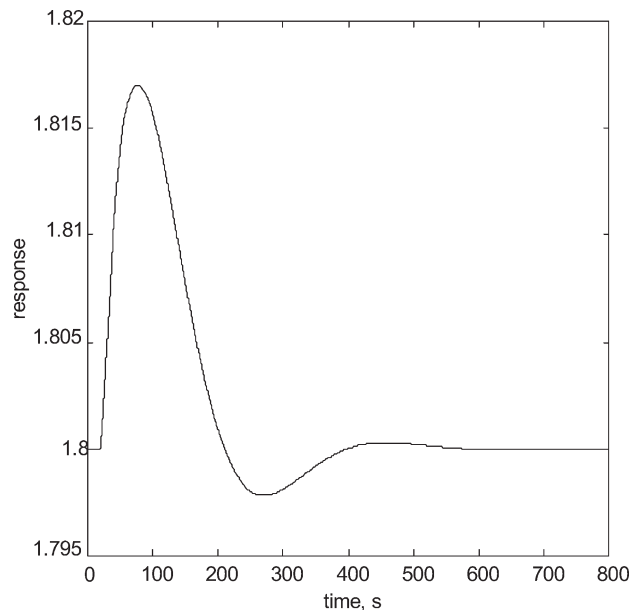

F i g . 6 – *Regulatory response of the non-ideal CSTR carrying out a reaction with rate $[-k_1\ c\ /(1 + k_2\ c)^2]$ $k_c = 1.2$, $\tau_I = 94.74$ ($\phi = 0.3$ and $\alpha = 1.93$) Q changed from $0.033 \cdot 10^{-3}$ to $0.03366 \cdot 10^{-3}$ m^3/s*

y_r – Set point

α – Ratio of coefficient of s in the numerator to that of denominator of closed loop transfer function

β – Set point weighting parameter

τ – Process time constant, s

τ_I – Integral time, s

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